Journal of Statistical Physics, Vol. 31, No. 2, 1983

Strange Attractors and Asymptotic Measures of Discrete-Time Dissipative Systems

D. Mayer[1,2] **and G. Roepstorff**[1]

Received November 9, 1982

We investigate asymptotic properties of certain discrete-time dynamical systems in two and three dimensions with solenoidal attractor. It is proved that the asymptotic measures, relevant for the generalized version of the ergodic theorem, all derive from one Haar measure.

KEY WORDS: Long time behavior of dissipative systems; asymptotic measures on strange attractors; correlation functions; Smale's solenoid.

INTRODUCTION

Recently there has been considerable interest in simple low-dimensional dynamical systems, either continuous or discrete. Sometimes these systems show features which mimic physical phenomena like turbulence. Presently, most of our understanding of the complicated behavior of nonlinear models rests on computer experiments or on the analysis of the Lorenz model.(1–3) As seems widely accepted, it is very hard to generally describe chaotic behavior by analytical methods.

In this paper we discuss a certain class of two-dimensional dissipative discrete-time models and give a detailed analytic description of their strange attractor. A prototype of these model systems was suggested by Kaplan and Yorke.(4) These authors performed computer experiments and used the results to support certain conjectures concerning the dimension of strange attractors in general.(5) Later Jensen and Oberman(6) studied the

[1] Institut für Theoretische Physik, RWTH Aachen, West Germany.
[2] Heisenberg-fellow

0022-4715/83/0500-0309$03.00/0

stochastic behavior of the same model and calculated time averages of simple observables by the path integral method. Problems involving the long-time behavior of dissipative systems call for some kind of ergodic theorem, in complete analogy, as one hopes, to conservative systems of statistical mechanics. The validity of an extended ergodic theorem for dissipative systems is, however, a very difficult question. The only systems which one presently understands fairly well are the Axiom-A systems. Here Bowen and Ruelle[7,8] proved a generalized ergodic theorem by showing that there exists a probability measure supported by the attractor and invariant under the dynamics, such that the expectation of an observable coincides with its time average.

But even for Axiom-A systems where existence is guaranteed it seems hopeless in most cases to find a simple analytic expression either for the measure or merely for the strange attractor supporting this measure. This is in sharp contrast to the situation in conventional statistical mechanics where Gibbs measures are abundant.

In the following we confine the discussion to a class of models that can be traced to a compact Abelian group known as the solenoid of van Dantzig.[9,10] In particular, the models as proposed by Kaplan and Yorke and a model by Smale[11,12] belong to this class. The appearance of a group in the context of attractors looks somewhat mysterious but is basic to our discussion. Using this device we are able to relate asymptotic properties of our systems to harmonic analysis on this group. We shall find that these models, though not Axiom-A, are derived from an Axiom-A system by simple geometric projections. So the result of Bowen and Ruelle applies and a generalized ergodic theorem holds in the sense that the time average of an observable coincides with its Haar integral on the solenoid. It follows from the work of Sinai[13] that we need only show that the conditional measure on the unstable manifold as obtained from the Haar measure is absolutely continuous with respect to Lebesgue measure. Most striking is the appearance of almost periodic functions on the real line which, to our knowledge, has been overlooked before. To each model in our class there is a (possibly vector-valued) almost periodic function h mapping the reals onto a dense subset of the attractor. It may be said that this function completely characterizes the asymptotic behavior of the system.

The paper is organized as follows: First, we introduce a simple two-dimensional dissipative discrete model:

$$x' = 2x^2 - 1, \qquad y' = x + \lambda y \qquad (0 < \lambda < 1)$$

Then we relate it to a model by Kaplan and Yorke and to a three-dimensional model of Smale. Using Williams' inverse limit construction we describe the strange attractor for the model of Smale in terms of a function

h on the solenoid S of van Dantzig. Knowing that the reals are densely embedded in S we restrict h to the reals and obtain

$$h(t) = \sum_{n=0}^{\infty} \lambda^n \exp(\pi i 2^{-n} t)$$

We show that this defines a Fourier series with respect to the group S. It follows rather easily that h is continuous on S and therefore is an almost periodic function for this group. As a consequence, the attractors reflect topological properties of the solenoid.

Next we discuss the Haar measure on the group S which is given by the mean value on its continuous almost periodic functions. The function h allows us to carry this measure to the attractors of the other models.

We briefly remind the reader that from a measure theoretic point of view the solenoid S is isomorphic to the $(1/2, 1/2)$ Bernoulli-shift implying strong stochastic properties for our two-dimensional models. Finally we use Sinai's characterization of the asymptotic measures on Axiom-A attractors to show that the above measures as derived from the Haar integral are indeed asymptotic measures.

We are very much indebted to Professor D. Ruelle, who suggested to us investigating the relation of our models to Smale's solenoid.

1. A SIMPLE MODEL

Let T be the transformation of the plane defined by

$$T(x, y) = (2x^2 - 1, x + \lambda y) \tag{1.1}$$

where $0 < \lambda < 1$. In contrast to the Henon map,[14] T is not invertible. Let us consider the strip $\Omega \subset \mathbb{R}^2$ defined by $|x| \leqslant 1$. Then Ω is invariant under T, and so is its complement Ω^c. Now, the orbit

$$(x_n, y_n) = T^n(x, y), \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

for initial values in Ω^c escapes to infinity, whereas, for $(x_0, y_0) \in \Omega$, the orbit is trapped by any of the compact sets

$$\Omega_\epsilon = \{(x, y) : |x| \leqslant 1, |y| \leqslant (1 - \lambda)^{-1} + \epsilon\}, \qquad \epsilon > 0 \tag{1.3}$$

This may easily be proved by an induction argument. Thus we restrict our attention to the domain Ω_0 containing the attractor A of the map T. Since

$$\Omega_0 \supset T\Omega_0 \supset T^2\Omega_0 \supset \cdots \supset A \tag{1.4}$$

we have that

$$A = \bigcap_{n \geqslant 0} T^n \Omega_0 \tag{1.5}$$

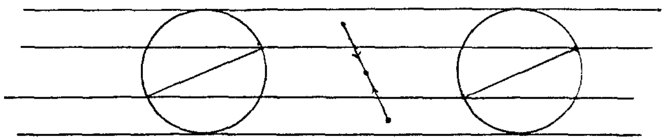

Fig. 1. The cylinder $\hat{\Omega}$ and the strip Ω inside it. The map p projects points from the cylinder to the strip.

2. A RELATED MODEL

The map (1.1) suggests changing coordinates,

$$x = \cos 2\pi u \tag{2.1}$$

and studying the related transformation

$$\hat{T}(u, y) = (2u, \lambda y + \cos 2\pi u) \tag{2.2}$$

of the cylinder $\hat{\Omega} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. The "chaotic" behavior of this model has been studied by Kaplan and Yorke.[4]

Because of the commutative diagram

$$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{p} & \Omega \\ \downarrow \hat{T} & & \downarrow T \\ \hat{\Omega} & \xrightarrow{p} & \Omega \end{array} \tag{2.3}$$

where $p(u, y) = (\cos 2\pi u, y)$, the two models are essentially the same. Notice, however, that p is a $2:1$ map since u and $1 - u$ are mapped onto the same x. The manifolds Ω and $\hat{\Omega}$ may be visualized in the three-dimensional Euclidean space of coordinates (x, y, z) where $x = \cos 2\pi u$ and $z = \sin 2\pi u$ such that Ω becomes part of the plane $z = 0$, i.e., the part inside the cylinder $x^2 + z^2 = 1$ which we identify with $\hat{\Omega}$ (see Fig. 1).

We obtain the attractor A of T by studying first the attractor $\hat{A}$ of $\hat{T}$ and then projecting it onto the plane: $A = p\hat{A}$. Let $\hat{\Omega}_0 = p^{-1}\Omega_0$. Then $\hat{T}\hat{\Omega}_0 \subset \hat{\Omega}_0$ and

$$\hat{A} = \bigcap_{n \geqslant 0} \hat{T}^n \hat{\Omega}_0 \tag{2.4}$$

3. A MODEL OF SMALE

Another related model is obtained if we replace the real variable y by a complex variable v writing

$$\tilde{T}(u, v) = (2u, \lambda v + e^{2\pi i u}) \tag{3.1}$$

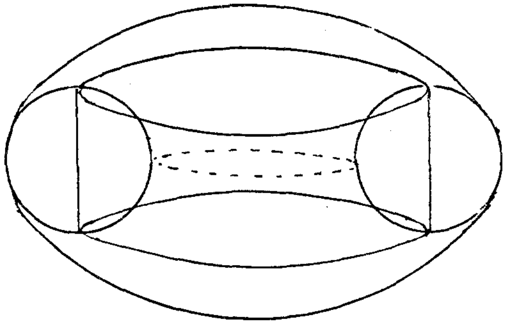

Fig. 2. The full torus wrapped around the finite cyclinder $\hat{\Omega}_0$. The map q projects points from the ring onto the cylinder. In particular, the attractor $\hat{A}$ on the cylinder is obtained by projecting the attractor of the Smale system.

to define a transformation $\tilde{T}$ of $\tilde{\Omega} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{C}$. Again, there is a commutative diagram

$$\begin{array}{ccc} \tilde{\Omega} & \xrightarrow{q} & \hat{\Omega} \\ \downarrow{\scriptstyle \tilde{T}} & & \downarrow{\scriptstyle \hat{T}} \\ \tilde{\Omega} & \xrightarrow[q]{} & \hat{\Omega} \end{array} \tag{3.2}$$

where the projection q is given by $(q(u,v) = (u, \mathrm{Re}\, v)$. The transformation (3.1) was introduced by Smale,[11],[3] who showed how its attractor may be identified with the dyadic solenoid of van Dantzig[9] provided $0 < \lambda < 1/2$. If $D \subset \mathbb{C}$ is the disk $|v| \leqslant (1-\lambda)^{-1}$, then the full torus $\tilde{\Omega}_0 = (\mathbb{R}/\mathbb{Z}) \times D$ contains the attractor $\tilde{A}$. Again, $\tilde{T}\tilde{\Omega}_0 \subset \tilde{\Omega}_0$ and $\tilde{A} = \bigcap_{n \geqslant 0} \tilde{T}^n \tilde{\Omega}_0$. The finite cylinder $\hat{\Omega}_0$ is embedded inside the full torus (see Fig. 2).

4. PREHISTORIES

Given a point ω_0 in $\tilde{\Omega}_0$ we would like to know all possible prehistories. If $\tilde{T}$ were invertible, then the prehistory of any point would be unique. However, with the map $u \mapsto 2u \pmod 1$ being $2:1$, there is a doubling of possible prehistories at each time step.

Following Williams[15,16] we describe the set of prehistories by the inverse (or projective) limit

$$X = \varprojlim X_n \tag{4.1}$$

[3] Smale uses coordinates (Θ, r, s) related to ours by the equations $\Theta = 2\pi u$, $r + is = (1-\lambda)v$.

where $X_n = \tilde{\Omega}_0$ for $n = 0, 1, 2, \ldots$ and where the limit is taken with respect to the following sequence of maps:

$$X_0 \xleftarrow{\tilde{T}} X_1 \xleftarrow{\tilde{T}} X_2 \xleftarrow{\tilde{T}} \cdots \tag{4.2}$$

Now the inverse limit construction is a common tool for investigating attractors. To be more specific, X is the set of all sequences $(\omega_0, \omega_1, \ldots)$ such that $\omega_n \in X_n$ and $\tilde{T}\omega_{n+1} = \omega_n$ for all n. There is a natural way in which $\tilde{T}$ induces a transformation $\tau : X \to X$, i.e.,

$$\tau(\omega_0, \omega_1, \omega_2, \ldots) = (\tilde{T}\omega_0, \omega_0, \omega_1, \ldots) \tag{4.3}$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \tilde{\Omega}_0 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tilde{T}} \\ X & \xrightarrow{\pi} & \tilde{\Omega}_0 \end{array} \tag{4.4}$$

where π is the projection onto the zeroth component:

$$\pi(\omega_0, \omega_1, \omega_2, \ldots) = \omega_0 \tag{4.5}$$

The significance of this construction is that τ can be inverted:

$$\tau^{-1}(\omega_0, \omega_1 \ldots) = (\omega_1, \omega_2, \ldots) \tag{4.6}$$

An immediate consequence is the invariance of the set πX:

$$\tilde{T}\pi X = \pi\tau X = \pi X \tag{4.7}$$

Now, any invariant subset of $\tilde{\Omega}_0$ has to be a subset of the attractor. Thus, $\pi X \subset \tilde{A}$. Even more is true for our construction:

$$\pi X = \tilde{A} \tag{4.8}$$

To have precise equality here is certainly a pleasant feature. It may be proved in the following way. Let X^n denote the subset of sequences $(\omega_0, \omega_1, \ldots) \in \prod_{k \geqslant 0} X_k$ satisfying $\omega_{k-1} = \tilde{T}\omega_k$ for $k = 1, \ldots, n$ $(n \geqslant 1)$. It is obvious that to any point $\omega_0 \in \tilde{T}^n\tilde{\Omega}_0$ there is a sequence $(\omega_0, \omega_1, \ldots) \in X^n$. Thus $\tilde{T}^n\tilde{\Omega}_0 \subset \pi X^n$. By definition of the inverse limit, $X = \bigcap_{n \geqslant 1} X^n$. This implies $\tilde{A} = \cap\, \tilde{T}^n\tilde{\Omega}_0 \subset \pi X$, which together with $\pi X \subset \tilde{A}$ proves (4.8).

We study the attractor via the space of prehistories, X, canonically associated with $\tilde{\Omega}_0$. It is important to realize that, in principle, a point on the attractor could have several prehistories. However, a little later we will show that $\pi : X \to \tilde{A}$ is bijective if $0 < \lambda < 1/2$.

5. THE SOLENOID

We proceed to give a detailed description of the space of prehistories, X. Let $(\omega_0, \omega_1, \ldots) \in X$ with $\omega_n = (u_n, v_n)$. From (3.1) and (4.2) we infer that

$$v_n = \lambda^{m-n} v_m + \sum_{k=1}^{m-n} \lambda^{k-1} \exp(2\pi i u_{n+k}) \tag{5.1}$$

for all $m > n$. Since $|\lambda| < 1$ and v_m is bounded, we obtain in the limit as $m \to \infty$

$$v_n = \sum_{k=0}^{\infty} \lambda^k \exp(2\pi i u_{n+k+1}) \tag{5.2}$$

This says that the prehistory of u_0 is sufficient to determine the prehistory of ω_0. All prehistories of the variable u are obtained by considering the inverse limit

$$S = \varprojlim S_n \tag{5.3}$$

where $S_n = \mathbb{R}/\mathbb{Z}$ for all $n \geqslant 0$ and where the map $S_n \leftarrow S_{n+1}$ means multiplication by 2. S is called the (dyadic) solenoid. Our analysis showed that the canonical projection $X \to S$, $\{u_n, v_n\} \mapsto \{u_n\}$ is bijective with inverse given by (5.2). More interesting, the combined map

$$\begin{gathered} S \to X \to \tilde{A} \\ \{u_n\} \mapsto \{u_n, v_n\} \mapsto (u_0, v_0) \end{gathered} \tag{5.4}$$

is bijective provided $0 < \lambda < 1/2$. This follows by an induction argument. Suppose u_n and v_n have already been determined from u_0 and v_0. We show how to construct u_{n+1} and v_{n+1}. From (5.2)

$$|v_n - e^{2\pi i u_{n+1}}| \leqslant \lambda/(1-\lambda) \tag{5.5}$$

There are two possibilities for u_{n+1}:

1. $\quad u_{n+1} = u_n/2, \qquad |v_n - e^{\pi i u_n}| \leqslant \lambda/(1-\lambda) \qquad (5.6)$

2. $\quad u_{n+1} = (u_n + 1)/2, \qquad |v_n + e^{\pi i u_n}| \leqslant \lambda/(1-\lambda) \qquad (5.7)$

Thus, v_n belongs to one of the two disks. The centers of these disks are a distance 2 apart. If $0 < \lambda < 1/2$, then $\lambda(1-\lambda)^{-1} < 1$ and the two disks do not intersect. Therefore, u_{n+1} is uniquely determined and so is

$$v_{n+1} = \lambda^{-1}(v_n - e^{2\pi i u_{n+1}}) \tag{5.8}$$

The preceding argument clearly shows that π is invertible if and only if $\tilde{T} : \tilde{A} \to \tilde{A}$ is invertible, independent of the details of the transformation

$\tilde{T}$. The situation may now be summarized as follows:

1. $0 < \lambda < 1/2$, $s \to \tilde{A}$ is bijective
2. $1/2 \leqslant \lambda < 1$, $s \to \tilde{A}$ is surjective

Thus, for λ smaller than $1/2$ the attractor of the Smale map is some sort of geometric picture in $\mathbb{R}^3$ of the abstract solenoid. This geometric realization of the dyadic solenoid was discovered first by van Dantzig[9] and with slight modifications by Stepanov and Tychonov.[12]

6. CONTINUITY

Addition modulo one gives S_n (which is $\mathbb{R}/\mathbb{Z}$) the structure of a complete topological group. As a subgroup of $\prod_{n \geqslant 0} S_n$, the solenoid S is endowed with the structure of a complete topological group as well. By Tychonov's theorem, this group is compact and Abelian. The topology of S is the weakest for which the canonical projections $S \to S_n$ are continuous.

Note that the real line is densely embedded in the solenoid since the equations

$$u_n = 2^{-n} t \pmod 1 \tag{6.1}$$

define a continuous injective homomorphism $\mathbb{R} \to S$, $t \mapsto \{u_n\}$ whose image is dense in S. Let $\mathbb{R}_*$ denote the reals with topology as subgroup of the solenoid. The topology of $\mathbb{R}_*$ is weaker than the usual one and S may be considered the completion of $\mathbb{R}_*$. In fact, S is a certain compactification of the real line (the Bohr compactification with respect to the module of continuous almost periodic functions having almost periods 2^n[17]) with uncountably many points "at infinity." A function $f: \mathbb{R}_* \to \mathbb{C}$ is continuous if and only if for any $\epsilon > 0$ there exist $\delta > 0$ and a positive integer n such that $2^{-n}|x - x'| \pmod 1 < \delta$ implies $|f(x) - f(x')| < \epsilon$, or equivalently if $|x - x'| < \delta$ implies

$$|f(x) - f(x' + m2^n)| < \epsilon \tag{6.2}$$

for all $m \in \mathbb{Z}$. Thus, the continuous functions on $\mathbb{R}_*$ are precisely those continuous functions on $\mathbb{R}$ that are almost periodic with almost periods 2^n.

Any continuous function on $\mathbb{R}_*$ may be uniquely extended to a continuous function on S. We are interested in the map $S \to \mathbb{C}$, $\{u_n\} \mapsto v_0$ which characterizes the attractor. Its restriction to $\mathbb{R}_*$ is the function[4]

$$h(t) = \sum_{n=0}^{\infty} \lambda^n \exp(\pi i 2^{-n} t) \tag{6.3}$$

[4] The function h satisfies the functional equation $h(2t) = h(t) + \exp(2\pi i t)$.

This function is reminiscent of the series

$$\sum_{n=0}^{\infty} \lambda^n \cos \mu^n t, \qquad \lambda\mu > 1 + 3\pi/2, \qquad \mu \text{ odd} \tag{6.4}$$

that Weierstrass[18] introduced to demonstrate the existence of a nowhere differentiable continuous function. We wish to show that the function $h(t)$ is continuous in a rather strong sense, i.e., with respect to the topology of $\mathbb{R}_*$. In (6.3) we encounter a concrete example of an absolutely convergent Fourier series with respect to the group S. Let us recall some notions from harmonic analysis and let us apply them in our case. A character on S is a continuous homomorphism $\chi: S \to U(1)$ where $U(1)$ is the multiplicative group of complex numbers with unit modulus. With respect to pointwise multiplication, the characters on S form a group $\hat{S}$, the dual of S. The topology of $\hat{S}$ is that of uniform convergence on compact sets of S. Since S is compact as a whole, $\hat{S}$ is a discrete Abelian group.

The next result states that the dual $\hat{S}$ of the solenoid is the additive group of dyadic rationals: Any character χ on S, when restricted to $\mathbb{R}_*$, is of the form

$$\chi(t) = \exp(2\pi i r t), \qquad r = 2^{-n}m \tag{6.5}$$

with $n, m \in \mathbb{Z}(n \geqslant 0)$ arbitrary. This may be shown as follows. The identity map $\mathbb{R} \to \mathbb{R}_*$ is continuous. Thus, any character on $\mathbb{R}_*$ is a character on $\mathbb{R}$, hence is of the form $\exp(2\pi i r t)$, $r \in \mathbb{R}$. The character is continuous on $\mathbb{R}_*$ iff it is continuous at $t = 0$, i.e., for $\epsilon > 0$ there are $\delta > 0$ and $n \geqslant 0$ such that $|t| < \delta$ implies

$$|e^{2\pi i r t} - e^{2\pi i r m 2^n}| < \epsilon \tag{6.6}$$

for all $m \in \mathbb{Z}$. We use the triangle inequality and get

$$|z^m - 1| < 2\epsilon, \qquad z = e^{2\pi i r 2^n} \tag{6.7}$$

If $\epsilon < 1$, $z = 1$ and thus $r2^n \in \mathbb{Z}$. Suppose a complex function f on the real line is represented as a series,

$$f(t) = \sum_{r \in \hat{S}} \hat{f}(r) e^{2\pi i r t} \tag{6.8}$$

Then $\hat{f}$ is called the *Fourier transform* of f. Note that the term "Fourier transform" is used here not in the traditional sense but rather with respect to the dual pair $(S, \hat{S})$. The sum (6.8) is well defined for every real t if $\hat{f} \in L^1(\hat{S})$, that is to say, if

$$\sum_{r \in \hat{S}} |\hat{f}(r)| < \infty \tag{6.9}$$

A standard and simple argument shows that f, as inverse transform of an L^1 function, is continuous on $\mathbb{R}_*$ and may thus be extended to a function on S. In particular, the function h as defined by the series (6.3) is continuous.

The extension of a character $\chi: \mathbb{R}_* \to U(1)$ is straightforward. With $r = 2^{-n}m$ we have that

$$\chi(u_0, u_1, \ldots) = \exp(2\pi i m u_n) \tag{6.10}$$

We recall the construction of the space of prehistories, X. It carries the topology of an inverse limit of spaces X_n, each homeomorphic to the full torus, $\tilde{\Omega}_0$. Note also that the attractor of the Smale map, $\tilde{A}$, is embedded in $\tilde{\Omega}_0$.

Lemma. If $0 < \lambda < 1$, the map $S \to X$, $\{u_n\} \mapsto \{u_n, v_n\}$ is a homeomorphism. In particular, the map

$$\sigma : S \to \tilde{A}, \qquad \{u_n\} \mapsto (u_0, v_0) \tag{6.11}$$

is continuous. If $0 < \lambda < 1/2$, then σ and $\tilde{T}: \tilde{A} \to \tilde{A}$ are homeomorphisms.

Proof. We already showed that the map $S \to X$ is bijective. Obviously, its inverse is continuous. It remains to prove that $S \to X$ is continuous. For this it is necessary and sufficient that, for any n, v_n is a continuous function of $(u_0, u_1, \ldots)$. Since v_n is given by an absolutely convergent Fourier series (5.2), it is indeed continuous. This proves the first part of the theorem. Assume now that $0 < \lambda < 1/2$. We already showed that σ is bijective. It remains to prove that σ^{-1} is continuous. If $\{u_n, v_n\} \in X$, then $(u_n, v_n) \in \tilde{A}$ for each n. The inverse of $\tilde{T}$ on $\tilde{A}$, $(u_n, v_n) \mapsto (u_{n+1}, v_{n+1})$ is given by the formulas (5.6)–(5.8). The construction involves two disks separated by a distance $2[1 - \lambda(1-\lambda)^{-1}]$ strictly greater than zero, so that u_{n+1} (and hence v_{n+1}) varies continuously with (u_n, v_n). By induction, $(u_0, v_0) \mapsto (u_n, v_n)$ is continuous for all n proving that σ^{-1} is continuous. The second part of this lemma was proved in Ref. 19 using a similar argument.

We conclude with some remarks about a class of models proposed by Kaplan and Yorke.[(4)] Suppose p is a complex function on the real line, periodic with period $1, p(u+1) = p(u)$. Let us consider a transformation of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{C}$ similar to the Smale map,

$$\begin{aligned} u' &= 2u \\ v' &= \lambda v + p(u) \end{aligned} \tag{6.12}$$

If p has an absolutely convergent Fourier series in the traditional sense,

$$p(u) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m u}, \qquad \sum_{m=-\infty}^{\infty} |c_m| < \infty \tag{6.13}$$

then most of our results carry over to this case. Again, there is a map from the solenoid onto the attractor that takes $\{u_n\}$ into (u_0, v_0), v_0 now being given by

$$v_0 = \sum_{m,n} c_m \lambda^n \exp(2\pi i m u_{n+1}) \tag{6.14}$$

where the sum is over $n, m \in \mathbb{Z}$ with $n \geqslant 0$. The next step would be to replace the equation (6.3) by

$$h(t) = \sum_{m,n} c_m \lambda^n \exp(\pi i m 2^{-n} t) \tag{6.15}$$

to obtain an absolutely convergent series representation for h.

7. CONNECTEDNESS

The solenoid S as the completion of $\mathbb{R}_*$ is connected. Thus $\tilde{A}$ as a continuous image of the solenoid is connected. Moreover, since the attractors $\hat{A}$ and A are continuous images of $\tilde{A}$, they are connected, too, which is not at all obvious if one looks at the computer pictures.[4] Also, a continuous image of the real line is dense in any of these attractors. To this we add the remark that the same is true for any attractor arising in the more complicated cases (6.12) provided (6.13) holds. However, the solenoid fails to be locally connected. Every neighborhood of a point in S contains a disconnected neighborhood homeomorphic to a direct product $I \times C$, where I is an interval and C is Cantor's triadic set. This may be demonstrated as follows. Let $\{a_n\}$ be any point in S. A fundamental system of neighborhoods consists of sets $U = U_n(\epsilon)$, $\epsilon < 1/2$. A point $(u_0, u_1, \ldots) \in S$ belongs to U if $|u_n - a_n| < \epsilon$. Denote this interval of real numbers by $I = I_n(\epsilon)$. A homeomorphism between U and $I \times C$ is set up as follows. To $(u_0, u_1, \ldots) \in U$ there corresponds $(u_n, c_n) \in I \times C$, where c_n is defined by its base 3 expansion,

$$c_n = \sum_{k=1}^{\infty} x_{n+k} 3^{-k} \tag{7.1}$$

with coefficients given by

$$x_k = \begin{cases} 0, & 0 \leqslant u_k < 1/2 \\ 2, & 1/2 \leqslant u_k < 1 \end{cases} \tag{7.2}$$

Recall that the Cantor discontinuum is the set of all members of the closed unit interval having a triadic expansion in which the digit one does not occur. Therefore, c_n belongs to C and any point in C arises via (7.1) from a point in U.

If we now combine these results with those of the last section, we obtain: If $0 < \lambda < 1/2$, the attractor $\tilde{A}$ has topological dimension one and is locally homeomorphic to an interval times the Cantor discontinuum. Though C has zero (one-dimensional) Lebesgue measure, there exist Cantor sets of nonzero Lebesgue measure. It is known that these sets might arise in a local structure analysis of attractors different from those discussed here.[23]

The situation becomes rather complicated if the attractor is not a homeomorphic image of the solenoid. For instance, the attractors $\hat{A}$ and A are merely locally homeomorphic to some quotient space $I \times C/\sim$ where $\sim$ is an equivalence relation induced by the map $S \to \hat{A}$ (respectively, A). Therefore, it remains unknown whether $\hat{A}$ and A are indeed locally disconnected, though with some effort one proves that each straight line $u = \text{const}$ meets $\hat{A}$ in a set homeomorphic to C. Supported by computer experiments, we conjecture that $\tilde{A}$, $\hat{A}$, and A are locally connected if $1/2 \leqslant \lambda < 1$ in contrast to the situation where $\lambda < 1/2$.

There are dynamical systems with globally disconnected attractors. For instance, the invertible transformation of the unit square I^2,

$$T(x, y) = \begin{cases} (2x, y/3), & 0 \leqslant x < 1/2 \\ (2x-1, (y+2)/3), & 1/2 \leqslant x < 1 \end{cases} \tag{7.3}$$

has $I \times C$ as its attractor.

8. THE HAAR MEASURE

For every compact group, one proves the existence and uniqueness of the normalized Haar measure. We shall devote this brief section to the construction of the Haar measure for the solenoid and shall demonstrate its significance later on. Suppose this measure—call it μ—has been constructed. Then, with regard to the chain of maps $S \to \tilde{A} \to \hat{A} \to A$, we obtain images of μ that describe certain distinguished measures for the dynamical systems considered. We denote these measures $\tilde{m}$, $\hat{m}$, and m. Each measure is supported by the corresponding attractor, hence it has rather large sets of measure zero and, for $\lambda < 1/2$, it is not absolutely continuous with respect to Lebesgue measure. However, these measures will turn out to be invariant and ergodic.

For any continuous function $f: \mathbb{R}_* \to \mathbb{C}$ we would write $\int f\, d\mu$ to denote the Haar integral. As is well known from the theory of almost periodic functions,[17] the Haar measure of the solenoid, μ, is given by the equation

$$\int f\, d\mu = \lim_{T \to \infty} T^{-1} \int_0^T f(t)\, dt \tag{8.1}$$

The argument is simple and runs as follows. Since f is continuous, it is almost periodic and the mean (8.1) exists. Standard arguments show that the mean is translationally invariant: (8.1) assigns the same mean value to f and f_a where $f_a(t) = f(t + a)$. This invariance already characterizes the Haar measure up to normalization. Obviously, $\int d\mu = 1$. Thus, the mean of a function coincides with its Haar integral.

As dynamical system, S has been assigned to a transformation $\tau : S \to S$ such that a step backwards in time shifts the sequence $\{u_n\} \in S$,

$$\tau^{-1}(u_0, u_1, \ldots) = (u_1, u_2, \ldots) \tag{8.2}$$

This simply reflects our interpretation of $\{u_n\}$ as the prehistory of the event u_0. A moment's reflection shows that τ is a continuous automorphism of the group S and may also be defined as the unique extension of

$$\tau : \mathbb{R}_* \to \mathbb{R}_*, \qquad t \mapsto 2t \tag{8.3}$$

Then the desired property of the measure μ, invariance under τ, follows directly from (8.1).

To write down the corresponding formulas for the mean of a function within the dynamical systems $(\tilde{\Omega}_0, \tilde{m}, \tilde{T})$, $(\hat{\Omega}_0, \hat{m}, \hat{T})$, and (Ω_0, m, T) is straightforward:

$$\int_{\tilde{A}} \tilde{f}\, d\tilde{m} = \lim_{T\to\infty} T^{-1} \int_0^T dt\, \tilde{f}\big(\bar{t}, h(t)\big), \qquad \tilde{f} \in C(\tilde{\Omega}_0) \tag{8.4}$$

$$\int_{\hat{A}} \hat{f}\, d\hat{m} = \lim_{T\to\infty} T^{-1} \int_0^T dt\, \hat{f}\big(\bar{t}, \operatorname{Re} h(t)\big), \qquad \hat{f} \in C(\hat{\Omega}_0) \tag{8.5}$$

$$\int_A f\, dm = \lim_{T\to\infty} T^{-1} \int_0^T dt\, f(\cos 2\pi t, \operatorname{Re} h(t)), \qquad f \in C(\Omega_0) \tag{8.6}$$

Here $\bar{t}$ denotes the fractional part of t. The invariance of these measures under $\tilde{T}$, $\hat{T}$, and T, respectively, follows from the commutative diagrams (2.3), (3.2), and (4.4).

For certain functions $f \in C(\Omega_0)$ including all polynomials the invariant mean can easily be computed:

$f(x, y)$	$\int_A f\, dm$
$x, x^n y \quad (n \geqslant 0)$	0
$2x^2, 4xy^2$	1
$2y^2$	$(1 - \lambda^2)^{-1}$

It is also straightforward to calculate correlation functions

$$\tilde{c}(k) = \int_{\tilde{A}} d\tilde{m}\, \tilde{f} \circ \tilde{T}^k \cdot \tilde{g} - \int_{\tilde{A}} \tilde{f}\, d\tilde{m} \int_{\tilde{A}} \tilde{g}\, d\tilde{m}, \qquad k \in \mathbb{N}$$

for functions $\tilde{f}, \tilde{g} \in C(\tilde{\Omega}_0)$:

$\tilde{f}(u,v)$	$\tilde{g}(u,v)$	$\tilde{c}(k)$
u	u	$2^{-k}/12$
v^*	v	$\lambda^k(1-\lambda^2)^{-1}$

We will come back to these results later. Explicit formulas of this type have first been obtained by Jensen and Oberman.[6]

9. EQUIVALENCE WITH A BERNOULLI SHIFT

The solenoid S which plays a fundamental role in our analysis has been equipped with the Haar measure μ and the automorphism τ. Disregarding topology and group structure, (S, μ, τ) is a measure space together with a measure preserving transformation, hence an abstract dynamical system. We remark that this system is in fact Bernoulli, though we feel this is commonplace to the expert. The first step in our argument is this. To any $(u_0, u_1, \ldots) \in S$ we assign a doubly infinite sequence $(\ldots u_{-1}, u_0, u_1, \ldots)$ where $u_{-n} = 2^n u_0 \in \mathbb{R}/\mathbb{Z}$ if $n > 0$. The new sequence describes both past and future of the event u_0 (a time step on $\mathbb{R}/\mathbb{Z}$ means multiplication by two). Now the transformation τ shifts the history in the obvious way

$$(\tau u)_n = u_{n-1}, \qquad u = (\ldots, u_{-1}, u_0, \ldots) \tag{9.1}$$

In a second step we define numbers a_n, $n \in \mathbb{Z}$,

$$a_n = \begin{cases} 0, & 0 \leqslant u_n < 1/2 \\ 1, & 1/2 \leqslant u_n < 1 \end{cases} \tag{9.2}$$

Note that $.a_0 a_{-1} a_{-2} a_{-3} \ldots$ is the base-2 expansion of u_0. For positive n, the numbers a_n uniquely determine the prehistory of u_0. We have thus found a correspondence between elements of S and doubly infinite sequences $(\ldots a_{-1}, a_0, a_1, \ldots)$ of binary digits. This correspondence is $1:1$ except for those sequences $\{a_n\}$ which eventually become constant to the left, that is to say if u_0 is a dyadic rational. They form a set of measure zero. Again,

$$(\tau a)_n = a_{n-1} \tag{9.3}$$

This is known as the two-sided Bernoulli $(1/2, 1/2)$-shift if

$$B = \prod_{n=-\infty}^{\infty} \{0, 1\} \tag{9.4}$$

is considered together with the direct product measure obtained from giving the two elements $\{0, 1\}$ equal probability,

$$p(0) = p(1) = 1/2 \tag{9.5}$$

We now show that this measure coincides with the Haar measure of the solenoid when transported to B. We argue that the Haar measure μ is uniquely determined by its invariance under translations by group elements. What is the group structure of B as induced by S? It is that of formal power series

$$a = \sum_{n=-\infty}^{\infty} a_n 2^n \tag{9.6}$$

It is understood that series are added as though they might represent real numbers. Notice that the inverse of a finite real non-negative number exists and is infinite. For instance, the inverse of the number 1 is

$$\sum_{n=0}^{\infty} 2^n \tag{9.7}$$

The product measure P is specified by its values on cylinders,

$$P\{a : a_n = r_n, n \in A\} = \prod_{n \in A} p(r_n) = 2^{-|A|} \tag{9.8}$$

where A is any finite subset of $\mathbb{Z}$ and $|A|$ is the number of elements in A. Call $|A|$ the diameter of the cylinder. Translation by a group element $b \in B$ maps cylinders onto cylinders. Obviously, such a translation preserves the diameter, hence preserves the measure. This would no longer be true for a Bernoulli $(p, 1-p)$-shift if $p \neq 1/2$. Suppose now that A is empty; then the cylinder is the entire space B and $P(B) = 1$ by (9.8). Thus P is the Haar measure on B. Since the lack of $1:1$ correspondance between the measure spaces S and B stems from sets of measure zero, we may still regard S and B as the "same" from a measure-theoretic point of view.

Since B and the attractor $\tilde{A}$ of the Smale map are related by a measure-theoretic isomorphism, the system $(\tilde{A}, \tilde{m}, \tilde{T})$ is Bernoulli. By contrast, the attractor $\hat{A}$ (and also A) is obtained from the Bernoulli system by a *noninvertible* map that transports the structure from B to $\hat{A}$ (respectively, A). In this case we do not know whether elimination of sets of measure zero can make B, $\hat{A}$, and A isomorphic. However, the systems $(\hat{A}, \hat{m}, \hat{T})$ and (A, m, T) are obviously mixing.

To see how $\mathbb{R}$ is embedded in B, we write

$$a(t) = t = \sum_{n=-\infty}^{\infty} a_n 2^n \tag{9.9}$$

for any $t \in \mathbb{R}_+$ and $a(t) = a(-t)^{-1}$ if $-t \in \mathbb{R}_+$. It then follows that $\mathbb{R}$ is mapped onto the sequences $(\ldots a_{-1}, a_0, a_1, \ldots)$ that eventually become constant to the right. Again, these sequences form a set of measure zero. Consequently, the subgroup $\mathbb{R}_*$ though dense in S has measure zero.

We recall the measure-theoretic isomorphism between the $(1/2, 1/2)$ Bernoulli system B and the unit square I^2 with planar Lebesgue measure. To the shift in B there corresponds the baker transformation.[20] This is a map of I^2 that expands by a factor of 2 the horizontal coordinate (the variable u_0 of the solenoid) while it contracts the vertical coordinate by a factor $1/2$. Let us now consider the conditional probability measure on the unstable manifold which is described by the interval $0 \leqslant u_0 < 1$, as induced by the planar Lebesgue measure and the σ-field of vertical strips in I^2. Then, as is well known, one simply recovers the one-dimensional Lebesgue measure for the variable u_0. This observation is essential to the final step of our argument.

10. ERGODICITY

We saw that the attractors of our simple dissipative systems can be explicitly described by the almost periodic function h as defined in Section 6. The same function allowed us to construct the measures (8.4) to (8.6). In this section we want to show that these measures describe the long-time behavior of our systems: the expectation value of any observable with respect to these measures equals its time average at least for almost all initial values in the basin of the attractor.

Let us first recall a theorem of Bowen and Ruelle[7,8] which solves this problem for Axiom-A systems:

Theorem. Let $T: M \to M$ be an Axiom-A diffeomorphism of the compact manifold M with a strange attractor A. Then there exists on A an unique T-invariant ergodic measure μ such that for almost all x in the basin $U(A)$ of A and for all continuous functions f on $U(A)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_A d\mu\, f = \langle f \rangle_\mu$$

If $c(k) = \langle g \circ T^k \cdot f \rangle_\mu - \langle g \rangle_\mu \langle f \rangle_\mu$ denotes the correlation function for the functions f and g in $U(A)$, then

$$|c(k)| \leqslant Ce^{-\beta k}$$

with some $\beta > 0$ and some constant C depending on f and g.

The asymptotic measure μ above can be characterized in different equivalent ways,[21] the most convenient for our purposes being the following:[13] μ is the unique T-invariant measure on A which induces conditional measures on its unstable manifolds absolutely continuous with respect to Lebesgue measure.

Unfortunately we cannot apply these results immediately to our models because none of them is Axiom-A. It is known, however, that model (3.1) of Smale can easily be extended to an Axiom-A system on the three-dimensional sphere with exactly the same strange attractor $\tilde{A}$.[22] Using the results of Section 9 on the conditional measures induced by $\tilde{m}$ on the unstable manifolds of $\tilde{A}$ we conclude that $\tilde{m}$ is indeed the asymptotic measure for the strange attractor $\tilde{A}$. Correlation functions therefore decay exponentially fast for this system.

It is now straightforward to prove similar properties for the measures $\hat{m}$ and m on the attractors $\hat{A}$ and A, respectively. Let us give the argument in the case of measure $\hat{m}$.

Let $\hat{U}(\hat{A}) = q(\tilde{\Omega}_0)$ be the projection of the solid torus $\tilde{\Omega}_0$ onto the cylinder $\hat{\Omega}$. It is a neighborhood in the basin of the attractor $\hat{A}$. Any continuous function $\hat{f}$ on $\hat{U}(\hat{A})$ defines a continuous function $\tilde{f}$ on $\tilde{\Omega}_0$ through $\tilde{f} = \hat{f} \circ q$. By the Bowen–Ruelle theorem we have for almost all $\tilde{x} \in \tilde{\Omega}_0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}(\tilde{T}^k \tilde{x}) = \int_{\tilde{A}} d\tilde{m}\, \tilde{f} \tag{10.1}$$

But this can be written as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}\left(\hat{T}^k (q\tilde{x})\right) = \int_{\hat{A}} d\hat{m}\, \hat{f} \tag{10.2}$$

where we used relation (3.2).

This being true for almost all $\tilde{x}$ in $\tilde{\Omega}_0$ we get for almost all $\hat{x}$ in $\hat{U}(\hat{A})$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}(\hat{T}^k \hat{x}) = \int_{A} d\hat{m}\, \hat{f} \tag{10.3}$$

which we wanted to show. The same argument applies also to the measure m on attractor A of system (1.1).

It is also easy to show that the correlation functions for both systems (1.1) and (2.2) decay exponentially fast, which agrees with our calculations in Section 8.

Because the measures $\tilde{m}$, $\hat{m}$, and m are explicitly known we can calculate all statistical properties of the above systems exactly.

We emphasize that everything in this section was stated under the provision that the parameter λ is smaller than $1/2$. What happens if $\lambda \geqslant 1/2$ is not at all clear. We conjecture that even then the Haar measure of S, via the continuous surjection $S \to \tilde{A}$, induces the correct asymptotic measure for the Smale model.

REFERENCES

1. E. N. Lorenz, Deterministic nonperiodic flow, *J. Atm. Sci.* **20**:130–141 (1963).
2. R. F. Williams, The structure of Lorenz attractors, in *Turbulence Seminar*, pp. 94–112. Lecture Notes in Mathematics 615 (Springer Verlag, New York, 1977).
3. R. D. Richtmyer, *Principles of Advanced Mathematical Physics*, Vol. II, Chap. 31 (Springer Verlag, New York, 1981).
4. J. L. Kaplan, and J. A. Yorke, Chaotic behavior of multidimensional difference equations, in *Functional Differential Equations and Approximations of Fixed Points*, pp. 228–237. Lecture Notes in Mathematics 730 (Springer Verlag, Berlin, 1979).
5. P. Frederickson, J. L. Kaplan, E. Yorke, and J. A. Yorke, The Liapunov dimension of strange attractors. Univ. of Maryland preprint (1981).
6. R. V. Jensen and C. R. Oberman, Calculation of the statistical properties of strange attractors, *Phys. Rev. Lett.* **46**:1547–1550 (1981).
7. D. Ruelle, A measure associated with Axiom-A attractors, *Am. J. Math.* **98**:619–654 (1976).
8. R. Bowen, and D. Ruelle, The ergodic theory of Axiom-A flows, *Inv. Math.* **29**:181–202 (1975).
9. D. Van Dantzig, Über topologisch homogene Kontinua, *Fund. Math.* **15**:102–125 (1930).
10. L. Vietoris, Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen, *Math. Annalen* **97**:454–472 (1927).
11. S. Smale, Dynamical systems and turbulence, in *Turbulence Seminar*, pp. 48–70, Lecture Notes in Mathematics 615 (Springer Verlag, New York, 1977).
12. W. Stepanov, and A. Tychonov, Über die Räume der fastperiodischen Funktionen. *Matem. Sbornik* **41**:166–178 (1934).
13. Ya. G. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surveys* **27**(4):21–69 (1972).
14. M. Henon, A two-dimensional mapping with a strange attractor, *Commun. Math. Phys.* **50**:69–77 (1976).
15. R. F. Williams, One dimensional non-wandering sets. *Topology* **6**:473–487 (1967).
16. R. F. Williams, A note on unstable homeomorphisms. *Proc. Am. Math. Soc.* **6**:308–309 (1955).
17. A. Weil, *L'integration dans les groupes topologiques et ses applications*. 2^{e} ed. Actual. Scient. et Ind., no. 869–1145, (Hermann, Paris, 1953).
18. K. Weierstrass, Über kontinuierliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen, in *Mathematische Werke von Karl Weierstrass*, Bd. 2, pp. 71–76 (Berlin, 1895). Reprinted by Georg Olms Verlagsbuchhandlung (Hildesheim).
19. S. E. Newhouse, Lectures in Dynamical Systems, in *Progress in Mathematics* 8, pp. 1–114 (Birkhäuser Verlag, Boston, 1980).
20. J. Moser, *Stable and Random Motions in Dynamical Systems*, Ann. of Math. Studies 77, pp. 62–64 (Princeton Univ. Press, 1973).
21. D. Ruelle, Differentiable dynamical systems and the problem of turbulence, *Bull. Am. Math. Soc.* **5**:29–42 (1981).
22. D. Ruelle, *Statistical Mechanics and Dynamical Systems*, Duke Univ. Math. Series III, pp. 1–108 (Duke University, Durham, North Carolina, 1977).
23. R. Bowen, A horseshoe with a positive measure, *Invent. Math.* **29**:203–204 (1975).